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The generalized Ornstein–Uhlenbeck process

Manuel O Cáceres†§ and Adrián A Budini‡

† Centro Atómico Bariloche and Instituto Balseiro, CNEA, and Universidad Nacional de Cuyo, Av. Ezquiel Bustillo Km 9.5, 8400 San Carlos de Bariloche, Río Negro, Argentina
‡ Universidad Nacional del Comahue, Dirección Postal: CAB, Av. Ezquiel Bustillo Km 9.5, 8400 San Carlos de Bariloche, Río Negro, Argentina

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Abstract. Langevin-like equations have been studied in the presence of arbitrary noise. The characteristic functional of the generalized Langevin process has been built up. Exact results for all cumulants are given. Particular stress has been put on the Campbell, dichotomous and radioactive decay noises. Transient relaxation, susceptibility and diffusion constants for different (noisy) media have been sketched in order to exemplify the theory. The generalized Ornstein–Uhlenbeck and Wiener processes have been completely characterized. The generalized Kubo oscillator has been worked out and all its 1-time moments have been calculated for different noise structures.

1. Introduction

Since the pioneering work by Ornstein and Uhlenbeck [1] the behaviour of systems under the effect of noise has attracted the interest of many workers. In particular (linear) Langevin-like equations (non-Markovian) for the relaxation of the velocity have been studied in different contexts. As a matter of fact the 2-state Brownons model [2] has captured the attention, for more than two decades, in order to describe problems such as noise-induced transitions [3–5], and Brownian motion in inhomogeneous mediums [6]. In a similar context, the generalized Wiener process driven by dichotomous [7] and white Poisson [8] noises have also been studied in the presence of non-natural boundary conditions.

It is well known that if the stochastic process (s.p.) $\boldsymbol{V}(t)$ is non-Markovian, a complete characterization of the s.p. $\boldsymbol{V}(t)$ demands the knowledge of the whole Kolmogorov hierarchy, i.e. the m-time joint probability distribution $P[V(t_1); V(t_2); \ldots; V(t_m)]$, or equivalently all the m-time moments $\langle V(t_1); V(t_2); \ldots; V(t_m)\rangle$. When partial knowledge of the s.p. is required, the 1-time probability distribution $P[V(t_1)]$ is enough. This is the case when only 1-time moments of the velocity $\langle V(t)^m\rangle$ are required [3, 5, 6]. This fact can easily be visualized using functional calculus, i.e. knowing $P[V(t_1)]$ is equivalent to knowledge of the characteristic function $\langle \exp \mathrm{i}kV(t_1)\rangle$; but in order to know the whole Kolmogorov hierarchy a knowledge of the characteristic functional $\langle \exp \int \mathrm{i}k(t)V(t)\,\mathrm{d}t\rangle$ is required, which of course is a much more complicated object [9]. In general if we knew the formal solution of the stochastic differential equation (for each realization of the noise) we could write an expansion for $\langle \exp \int \mathrm{i}k(t)V(t)\,\mathrm{d}t\rangle$ in terms of the cumulants of the noise. This is not our task here because we are interested in a closed expression for this functional.

§ E-mail address: caceres@cab.cnea.edu.ar

Here we are concerned with obtaining a complete characterization of the generalized Ornstein–Uhlenbeck process—with natural boundary conditions—i.e. when the noise term is an arbitrary s.p. $\boldsymbol{\xi}(t)$. The case when the coefficients are time dependent has also been worked out. Therefore by using the characteristic functional of the s.p. $\boldsymbol{V}(t)$ it will be straightforward to calculate any m-time moment. We have exemplified the method with the calculation of linear relaxation mechanisms, correlation functions, and diffusion constants driven by different (additive) noise structures (Campbell, dichotomous, radioactive decay, Poisson, and in general any non-Gaussian noise). This approach is exact and provides a systematic starting point to obtain higher-order cumulants. We should comment that this method may be considered as complementary to a direct calculation of the moments from the generalized Langevin-like equation. Nevertheless proposition 1 (or 2), below, is more than that because it provides a systematic way of calculating non-trivial object such as $\langle \cos \int^t V(s)\, \mathrm{d}s \rangle$, etc. On the other hand the possibility of having a closed expression for the characteristic functional of the generalized Ornstein–Uhlenbeck process allows us to find all the 1-time moments of a particular class of multiplicative stochastic differential equations. We will demonstrate this fact with propositions 4 and 5 below.

The generalized Wiener process $\boldsymbol{X}(t)$—with natural boundary conditions—has also been worked out and their complete characterization has been given in terms of its characteristic functional (proposition 3).

A multiplicative model, the so-called Kubo oscillator [10] $\boldsymbol{U}(t)$ with an arbitrary noise has been worked out. In this case we cannot calculate (in a closed way) the characteristic functional of the s.p. $\boldsymbol{U}(t)$, nevertheless we succeed in obtaining all the 1-time moments $\langle U(t)^m \rangle$. We also present a formal expression for the 1-time probability distribution of the process $\boldsymbol{U}(t)$. The analysis of the complex Kubo oscillator has also been done for three different noises.

Some (particular) nonlinear stochastic differential equations (with arbitrary noise) are shown to be reduced—in a similar way—in terms of our previous propositions.

2. The generalized Ornstein–Uhlenbeck process

The equation of motion of a one-dimensional Brownian particle in a generalized medium has the Langevin-like form

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\gamma V + \xi(t) \qquad V \in [-\infty, \infty] \tag{2.1}$$

where V is the particle velocity, $\gamma > 0$ the friction constant divided by the mass of the particle, and $\xi(t) \in \mathrm{Re}$ is an arbitrary time-dependent random force characterizing the medium (*the noise*). When the s.p. $\boldsymbol{\xi}(t)$ is a zero-mean Gaussian white noise of intensity $\langle \xi(t_1)\xi(t_2)\rangle = 2\gamma k_\mathrm{B} T \delta(t_1 - t_2)$, equation (2.1) is the usual Langevin equation. Thus, process $\boldsymbol{V}(t)$ characterizes the velocity of a Rayleigh particle, so in this limit equation (2.1) is the Ornstein–Uhlenbeck process [9].

Let us generalize the noise $\xi(t)$, with $t \in [0, \infty]$, to any arbitrary s.p. characterized by the characteristic functional

$$G_\xi([k(t)]) \equiv \left\langle \exp \int_0^\infty \mathrm{i}k(t)\xi(t)\, \mathrm{d}t \right\rangle. \tag{2.2}$$

The notation $G_\xi([k(t)])$ emphasizes that G depends on the whole function $k(t)$ not just on the value it takes at one particular time. The convergence of the integral is accomplished because the functions $k(t)$ may be restricted to those that vanish for sufficiently large t.

Proposition 1. The characteristic functional of the s.p. $\boldsymbol{V}(t)$, with a sure (non-random) initial condition $V(0) = V_0$ (the case when V_0 is a random variable is a trivial generalization†), is

$$G_V([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0V_0} G_\xi\left(\left[\mathrm{e}^{\gamma t}k_0 - \int_0^t \mathrm{e}^{\gamma(t-s)} Z(s)\,\mathrm{d}s\right]\right) \tag{2.3}$$

where k_0 is a functional of $Z(t)$ given by

$$k_0 = \int_0^\infty Z(s)\exp(-\gamma s)\,\mathrm{d}s. \tag{2.4}$$

Proof. The proof follows by integrating by parts, and from the fact that (2.1) can be written as $\xi(t) = \dot{V} + \gamma V$. Thus the characteristic functional of noise $\xi(t)$ is connected to the characteristic functional s.p. $\boldsymbol{V}(t)$ by the relation

$$G_\xi([k(t)]) = \mathrm{e}^{-\mathrm{i}k_0V_0}\ G_V\left(\left[-\mathrm{e}^{+\gamma t}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\gamma t}k(t)\right]\right).$$

Defining the test function $Z(t) = -\mathrm{e}^{+\gamma t}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\gamma t}k(t)$ we get (2.3). □

Therefore for any arbitrary noise $\xi(t)$ (characterized by its functional $G_\xi([k(t)])$) all the m-time moments of the s.p. $\boldsymbol{V}(t)$ follow from mth order functional differentiation (often called the variation or Fréchet-derivative), i.e.

$$\langle V(t_1)V(t_2)\ldots V(t_m)\rangle = \mathrm{i}^{-m}\frac{\delta}{\delta Z(t_1)}\cdots\frac{\delta}{\delta Z(t_m)}G_V([Z(t)])\bigg|_{Z=0}. \tag{2.5}$$

Also, all cumulants of the s.p. $\boldsymbol{V}(t)$ follow from their cluster properties, which are sketched in appendix A.

There are *simple* noises which can easily be handled by using the cumulant-expansion technique, it is therefore straightforward to find their functional. These are the cases of Gaussian non-white noises and non-Gaussian white noises [9] (see appendix B). So using proposition 1 and (B1) (for the Gaussian non-white noise), or (B3) (for the non-Gaussian white noise) all moments of the s.p. $\boldsymbol{V}(t)$ follow immediately.

2.1. The Campbell model for the noise $\xi(t)$

Following proposition 1, to characterize the s.p. of the velocity $\boldsymbol{V}(t)$ we need to know the noise functional $G_\xi([k(t)])$. Unfortunately there are only a few s.p. which can be summed in order to find $G_\xi([k(t)])$ in a closed way, of which Campbell's is one [11].

A stochastic realization, with $t \in [0, \infty]$, of a generalized Campbell process is given by ‡

$$\xi(t) \equiv \sum_{\sigma=1}^{s}\psi(t - t_\sigma) \tag{2.6}$$

where for each random integer s there is a set of independent random times obeying $\{t_1 < t_2 < \ldots < t_s\}$ (ordered equally distributed independent random dots), and $\psi(t)$ is a given function with finite support. Then, the characteristic functional of s.p. $\boldsymbol{\xi}(t)$, in the interval $[0, \infty]$, is

$$G_\xi([k(t)]) = \exp\int_0^\infty\left(\exp\left[\mathrm{i}\int_0^\infty k(t)\psi(t-\tau)\,\mathrm{d}t\right] - 1\right)q(\tau)\,\mathrm{d}\tau \tag{2.7}$$

where $q(\tau)$ is the density of one dot.

† The average over the initial condition V_0 can trivially be taken at the end of all the calculations.

‡ See chapter II of van Kampen's book [9].

Application. The generalized Ornstein–Uhlenbeck process, with Campbell's noise, is completely characterized by the functional (with $t \in [0, \infty]$)

$$G_V([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0 V_0} \exp \int_0^\infty \left(\exp\left[\mathrm{i} \int_0^\infty \int_t^\infty \mathrm{e}^{\gamma(t-s)} Z(s)\, \mathrm{d}s \;\; \psi(t-\tau)\, \mathrm{d}t \right] - 1 \right) q(\tau)\, \mathrm{d}\tau \tag{2.8}$$

where k_0 is given by (2.4). This follows immediately from (2.3) and (2.7).

Examples. From (2.8) all the m-time moments of the s.p. $\boldsymbol{V}(t)$ follow by taking functional derivative of $G_V([Z(t)])$. For example the 1- and 2-time moments are

$$\langle V(\mu)\rangle = \mathrm{i}^{-1} \frac{\delta}{\delta Z(\mu)} G_V([Z(t)]) \bigg|_{Z=0} = \mathrm{e}^{-\gamma\mu} \left\{ V_0 + \int_0^\infty \mathrm{d}\tau\, q(\tau) \int_0^\mu \mathrm{e}^{\gamma s} \psi(s-\tau)\, \mathrm{d}s \right\} \tag{2.9}$$

$$\begin{aligned}\langle V(\alpha)V(\mu)\rangle = \mathrm{i}^{-2} \frac{\delta}{\delta Z(\alpha)} \frac{\delta}{\delta Z(\mu)} G_V([Z(t)]) \bigg|_{Z=0} &= \mathrm{e}^{-\gamma(\alpha+\mu)} \bigg\{ V_0^2 + V_0 \int_0^\infty \mathrm{d}\tau\, q(\tau) \\ &\times \left[\int_0^\alpha \mathrm{e}^{\gamma s} \psi(s-\tau)\, \mathrm{d}s + \int_0^\mu \mathrm{e}^{\gamma s} \psi(s-\tau)\, \mathrm{d}s \right] \\ &+ \left[\int_0^\infty \mathrm{d}\tau\, q(\tau) \int_0^\alpha \mathrm{e}^{\gamma s} \psi(s-\tau)\, \mathrm{d}s \right] \left[\int_0^\infty \mathrm{d}\tau'\, q(\tau') \int_0^\mu \mathrm{e}^{\gamma s'} \psi(s'-\tau')\, \mathrm{d}s' \right] \\ &+ \int_0^\infty \mathrm{d}\tau\, q(\tau) \int_0^\alpha \mathrm{e}^{\gamma s} \psi(s-\tau)\, \mathrm{d}s \int_0^\mu \mathrm{e}^{\gamma s'} \psi(s'-\tau)\, \mathrm{d}s' \bigg\}. \end{aligned} \tag{2.10}$$

Obviously the time-dependent behaviour of the moments of the s.p. $\boldsymbol{V}(t)$ will depend on the shape of the pulse $\psi(\tau)$ and on the density $q(\tau)$. From (2.9) and (2.10) the explicit correlation function of the s.p. $\boldsymbol{V}(t)$ is

$$\langle\langle V(\mu)V(\alpha)\rangle\rangle = \int_0^\alpha \mathrm{d}y \int_0^\mu \mathrm{d}x\, \mathrm{e}^{-\gamma(x+y)} \langle\langle \xi(\alpha-x)\xi(\mu-y)\rangle\rangle \tag{2.11}$$

where

$$\langle\langle \xi(\alpha-x)\xi(\mu-y)\rangle\rangle = \int_0^\infty q(\tau)\psi(\alpha-x-\tau)\psi(\mu-y-\tau)\, \mathrm{d}\tau$$

is the correlation function of the noise.

The half-Fourier transform of the stationary 2-time cumulant of the s.p. $\boldsymbol{V}(t)$ is related to the susceptibility of the system, so (2.11) provides a systematic way of calculating susceptibilities for different Campbell's noise models.

If the density of one dot is $q(t) = \exp(-t/T)$ and the pulse is $\psi(t) = \delta(t)$ the Campbell's noise corresponds to a non-stationary white-noise process similar to (B2). However in this case the cumulants of this process follow by functional differentiation of $\ln G_\xi([k(t)]) = \sum_{m=1}^\infty \frac{\mathrm{i}^m}{m!} \int_0^\infty q(t)k(t)^m\, \mathrm{d}t$. Note that this noise can be related to a particular decay model.

If the density $q(\tau)$ is uniform $q(\tau) = \nu/T$ with $\tau \in [0, T]$, in the limit $T \to \infty$ and $\nu \to \infty$ with $\nu/T =$ constant, we can define $\rho = \nu/T$ which represents the average number of events per unit of time, thus $\xi(t)$ is a non-white Campbell's noise with pulses $\psi(t)$.

When $\psi(t) = A^2 \exp(-|t|/\tau_c)$, from (2.9) we get for the transient behaviour of the moment of the s.p. $\boldsymbol{V}(t)$

$$\langle V(\mu)\rangle = \mathrm{e}^{-\gamma\mu} V_0 + A^2 \rho \tau_c \left\{ \frac{2}{\gamma}[1 - \mathrm{e}^{-\gamma\mu}] - \frac{\mathrm{e}^{-\gamma\mu}}{(\gamma - \tau_\mathrm{c}^{-1})} [\mathrm{e}^{(\gamma-\tau_\mathrm{c}^{-1})\mu} - 1] \right\} \tag{2.12}$$

The third term of (2.12) slows down when the correlation length of the noise (τ_c) is equal to γ^{-1} (the dissipation parameter)†. The stationary correlation function (i.e. $\alpha, \mu \to \infty$ but $\alpha - \mu \neq 0$) is

$$\langle\langle V(\mu)V(\alpha)\rangle\rangle_{st} = \rho A^4 \left\{ \frac{1}{2\gamma} e^{-\gamma|\mu-\alpha|} \left[\frac{1}{(\gamma + \tau_c^{-1})} - \frac{1}{(\gamma - \tau_c^{-1})} \right]^2 + e^{-|\mu-\alpha|/\tau_c} \left[\frac{|\mu-\alpha|}{(\gamma + \tau_c^{-1})(\gamma - \tau_c^{-1})} \right] + e^{-|\mu-\alpha|/\tau_c} \left[\frac{\tau_c}{(\gamma^2 - \tau_c^{-2})} - \frac{2\tau_c^{-1}}{(\gamma^2 - \tau_c^{-2})^2} \right] \right\} \tag{2.13}$$

which is well defined for any γ and τ_c. Therefore this model shows, in the correlation of the s.p. $\boldsymbol{V}(t)$, a non-exponential contribution. Campbell's noise conduces to the fact that the correlation (2.13) will show a longer relaxation tail when $\gamma \sim \tau_c$. This is an interesting phenomenon to consider when the susceptibility is the important object to be analysed‡.

The zero-frequency susceptibility is related to the diffusion constant, thus for this model we get

$$D \equiv \int_0^\infty \langle\langle V(t)V(0)\rangle\rangle_{st}\, dt = 2\rho A^4 \tau_c^2/\gamma^2.$$

For Campbell's white noise (shot noise), i.e. when $\psi(t) = B\delta(t)$, the diffusion constant is given by $D = \frac{B^2\rho}{2\gamma^2}$, which is an expected result if we identify $\Gamma_2 = \frac{1}{2}B^2\rho$ (see appendix B).

2.2. *The dichotomous model for the noise $\xi(t)$*

The dichotomous s.p. is a 2-state Markov process defined by a 2×2 master operator, which is completely characterized by the evolution equation of the conditional probability $P(\xi(t)|\xi(t_0))$. In general the random variable ξ can take two arbitrary values, but here we will restrict the analysis to the symmetric case, i.e. $\xi = \pm a$, and adopt equal 'up' and 'down' hopping transition rates λ.

Denoting $\boldsymbol{P}$ as a two-dimensional vector, the master equation for the conditional probability can be written as

$$\dot{\boldsymbol{P}} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix} \boldsymbol{P} \tag{2.14}$$

with the initial condition $\boldsymbol{P}(t=0) = \boldsymbol{\delta}_{\xi;\xi_0}$.

For the dichotomous noise we cannot get a closed expression for its characteristic functional, thus we give its series expansion (see (C7)).

Application. Following proposition 1, the generalized Ornstein–Uhlenbeck process, with dichotomous noise, is completely characterized by the functional (with $t \in [0, \infty]$)

$$G_V([Z(t)]) = e^{+iV_0\int_0^\infty e^{-\gamma s} Z(s)\, ds} \left\{ 1 - a^2 \int_0^\infty dt_1 \int_0^{t_1} dt_2 k(t_1)k(t_2) e^{-2\lambda(t_1-t_2)} + a^4 \times \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_4} dt_4 k(t_1)k(t_2)k(t_3)k(t_4) e^{-2\lambda(t_1-t_2+t_3-t_4)} + \cdots \right\}$$

† A similar situation occurs when a dichotomous noise is used [5].

‡ Introducing two statistically independent Campbell's processes $\boldsymbol{\xi}_1(t)$, $\boldsymbol{\xi}_2(t)$ such that $\langle\xi(t)\rangle = \langle\xi_1(t) - \xi_2(t)\rangle = 0$, and in the limit μ, α going to infinity, the first moment of the s.p. $\boldsymbol{V}(t)$ is zero and the second cumulant is equivalent to (2.13).

$$+\mathrm{i}\left\{a\int_0^\infty \mathrm{d}t_1 k(t_1)\mathrm{e}^{-2\lambda t_1} - a^3\int_0^\infty \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 \times\int_0^{t_2}\mathrm{d}t_3 k(t_1)k(t_2)k(t_3)\mathrm{e}^{-2\lambda(t_1-t_2+t_3)} + a^5\int_0^\infty \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3\int_0^{t_3}\mathrm{d}t_4 \times\int_0^{t_4}\mathrm{d}t_5 k(t_1)k(t_2)k(t_3)k(t_4)k(t_5)\mathrm{e}^{-2\lambda(t_1-t_2+t_3-t_4+t_5)} - \cdots\right\}(\delta_{a,\xi_0} - \delta_{-a,\xi_0}) \tag{2.15}$$

where

$$k(t) = \int_t^\infty \mathrm{e}^{\gamma(t-s)} Z(s)\,\mathrm{d}s. \tag{2.16}$$

This is accomplished from (2.3), (2.4) and the series expansion of the functional of the noise (C7).

Examples. All the m-time moments of s.p. $\boldsymbol{V}(t)$ (with dichotomous noise) follow by functional differentiation of (2.15). The first moment is

$$\langle V(\mu)\rangle = \mathrm{i}^{-1}\frac{\delta}{\delta Z(\mu)}G_V([Z(t)])\bigg|_{Z=0} = V_0\mathrm{e}^{-\gamma\mu} - \frac{\xi_0\mathrm{e}^{-\gamma\mu}}{(2\lambda-\gamma)}[\mathrm{e}^{-(2\lambda-\gamma)\mu} - 1] \tag{2.17}$$

from which we can appreciate that for $2\lambda = \gamma$ there is a similar slowing down (in the linear relaxation) as with Campbell's noise.

The 2-time moment gives

$$\langle V(\alpha)V(\mu)\rangle = -\frac{\delta}{\delta Z(\alpha)}\frac{\delta}{\delta Z(\mu)}G_V([Z(t)])\bigg|_{Z=0} = V_0^2\mathrm{e}^{-\gamma(\alpha+\mu)} - \frac{V_0\xi_0\mathrm{e}^{-\gamma(\alpha+\mu)}}{(2\lambda-\gamma)}[\mathrm{e}^{-(2\lambda-\gamma)\alpha} + \mathrm{e}^{-(2\lambda-\gamma)\mu} - 2] + \frac{2a^2}{(2\lambda+\gamma)}\left\{\frac{\mathrm{e}^{-\gamma|\alpha-\mu|} - \mathrm{e}^{-\gamma(\alpha+\mu)}}{2\gamma} + \frac{\mathrm{e}^{-\gamma\max(\alpha;\mu)-2\lambda\min(\alpha;\mu)} - \mathrm{e}^{-\gamma(\alpha+\mu)}}{(2\lambda-\gamma)}\right\}. \tag{2.18}$$

In the particular case $\alpha = \mu$, $V_0 = 0$ (and in the stationary limit) (2.18) reduces to the result found by Morita [5]. From (2.17), (2.18) and using a half-Fourier transform it is straightforward to find the susceptibility. Note that the stationary correlation function of the s.p. $\boldsymbol{V}(t)$ with Campbell's noise (2.13) leads to a non-Lorentzian susceptibility, a situation which does not occur for (2.18).

In this case the diffusion constant is $D = \frac{a^2}{\gamma^2(2\lambda+\gamma)}$. This result is in complete agreement with Gitterman's model [6], when the particle is only allowed to move inside the cluster†.

We remark that (2.15) is an exact result and gives us a starting point to find the susceptibility and more complex objects like higher-order cumulants. It is also possible to see from the functional (2.15) that the characteristic function of the 1-time probability density $P[V(t_1)]$ can be found by using the test function $Z(t) = k\delta(t - t_1)$ in agreement with Morita's momenta.

† To see this, note that from (2.18) the stationary mean-square velocity gives: $\langle V^2\rangle_{\mathrm{st}} = \frac{a^2}{(2\lambda+\gamma)\gamma}$, a result which was pointed out in equation (4) and (24) of Gitterman's paper [6] (in the appropriate limit).

2.3. The radioactive-decay model for the noise $\xi(t)$

Let us consider the noise as a constant function of time, which decreases by finite steps at random times t_i. If the probability per unit time for such steps is a constant β, the noise can be described by a one-step master equation [9]. This noise can be related with the number $N(t)$ of active nuclei surviving at time $t > 0$, which is a non-stationary Markov process. Therefore a *radioactive decay noise* is modelled by a Markov chain (with an absorbing site: $\xi = 0$) having a discrete range $\xi = 0, 1, 2, 3 \ldots$, initial value ξ_0 and master equation:

$$\frac{1}{\beta}\partial_t T(\xi|\xi_0) = (\xi+1)T(\xi+1|\xi_0) - \xi T(\xi|\xi_0). \tag{2.19}$$

When the *random force* in (2.1) is modelled by this type of non-stationary noise, we might consider its functional which can be obtained by using the Darling–Siegert theorem [12] (see appendix C) and is given [13] by

$$G_\xi([k(t)]) = \left[\beta\int_0^\infty \exp\left(-t\beta + \mathrm{i}\int_0^t k(s)\,\mathrm{d}s\right)\mathrm{d}t\right]^{\xi_0}. \tag{2.20}$$

From this functional it is simple to see that the first moment and the correlation function of the s.p. $\boldsymbol{\xi}(\mu)$ are

$$\langle\xi(\mu)\rangle = \xi_0 \mathrm{e}^{-\beta\mu}$$
$$\langle\langle\xi(\mu)\xi(\alpha)\rangle\rangle = \xi_0(\mathrm{e}^{-\beta\max(\mu,\alpha)} - \mathrm{e}^{-\beta(\mu+\alpha)}).$$

Application. The generalized Ornstein–Uhlenbeck process, with radioactive decay noise, is completely characterized by the functional (with $t \in [0,\infty]$)

$$G_V([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0V_0}\left[\beta\int_0^\infty \mathrm{d}t\,\exp\left(-t\beta + \mathrm{i}\int_0^t\left(\int_{t'}^\infty \mathrm{e}^{\gamma(t'-s)}Z(s)\,\mathrm{d}s\right)\mathrm{d}t'\right)\right]^{\xi_0} \tag{2.21}$$

where k_0 is given in (2.4). This fact follows immediately from (2.3) and (2.20).

Examples. From (2.21) all the m-time moments of s.p. $\boldsymbol{V}(t)$ follow by taking the functional derivative of $G_V([Z(t)])$. The 1-time and 2-time moment are

$$\langle V(\mu)\rangle = \mathrm{e}^{-\gamma\mu}V_0 + \frac{\xi_0\beta}{\gamma}\mathrm{e}^{-\gamma\mu}\left[\frac{1-\mathrm{e}^{-(\beta-\gamma)\mu}}{(\beta-\gamma)} + \frac{1}{\beta}\mathrm{e}^{-(\beta-\gamma)\mu} - \frac{1}{\beta}\right] \tag{2.22}$$

$$\begin{aligned}\langle\langle V(\mu)V(\alpha)\rangle\rangle = {}& -\beta^2\xi_0\frac{\mathrm{e}^{-\gamma(\mu+\alpha)}}{\gamma^2}\left[\frac{1-\mathrm{e}^{-(\beta-\gamma)\mu}}{(\beta-\gamma)} + \frac{1}{\beta}\mathrm{e}^{-(\beta-\gamma)\mu} - \frac{1}{\beta}\right]\\ &\times\left[\frac{1-\mathrm{e}^{-(\beta-\gamma)\alpha}}{(\beta-\gamma)} + \frac{1}{\beta}\mathrm{e}^{-(\beta-\gamma)\alpha} - \frac{1}{\beta}\right] + \beta\xi_0\frac{\mathrm{e}^{-\gamma(\mu+\alpha)}}{\gamma^2}\left[\frac{1-\mathrm{e}^{-(\beta-2\gamma)\min(\mu,\alpha)}}{(\beta-2\gamma)}\right.\\ &\left.+2\frac{\mathrm{e}^{-(\beta-\gamma)\min(\mu,\alpha)}-1}{(\beta-\gamma)} - \frac{\mathrm{e}^{-\beta\min(\mu,\alpha)}-1}{\beta}\right]\\ &+\xi_0\frac{\mathrm{e}^{-\beta\min(\mu,\alpha)}}{\gamma^2}(1-\mathrm{e}^{-\gamma\mu})(1-\mathrm{e}^{-\gamma\alpha}).\end{aligned} \tag{2.23}$$

For γ near to β (or $\frac{\beta}{2}$) these expressions show a non-trivial transient regime, which is similar to the one appearing with the Campbell and the dichotomous noises. In this particular model, the long-time limit of the correlation function is zero [i.e. $\alpha, \mu \to \infty$ but $\alpha - \mu \neq 0$] as expected.

3. The time-dependent linear case

If a linear stochastic differential equation has time-dependent coefficients, the same functional approach can be used to tackle the problem. A time-dependent linear case could arise when using the $\Omega-$expansion [9] in nonlinear stochastic differential equations. For example it might be necessary to analyse the fluctuation spectrum of a nonlinear system driven by a quasi-Gaussian noise (or maybe some non-white noise).

Using the $\Omega-$expansion it is well known that a *linear noise approximation* corresponds to the Gaussian approximation, thus we propose here to study the following time-dependent (linear) stochastic differential equation

$$\dot{Y} = -\gamma_1(t)Y + \gamma_2(t)\xi(t) \qquad Y \in [-\infty, \infty] \tag{3.1}$$

where the *random force* is any process completely characterized by its functional $G_\xi([k(t)])$, $\gamma_2(t)$ is a sure function of time, and $\gamma_1(t)$ is a positive function of time†. Thus the complete characterization of the s.p. $\boldsymbol{Y}(t)$ follows by

Proposition 2. The characteristic functional of the s.p. $\boldsymbol{Y}(t)$ is

$$G_Y([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0Y_0}\ G_\eta\left(\left[e^{\int_0^t \gamma_1(s)\,\mathrm{d}s}k_0 - \int_0^t \mathrm{e}^{\int_{t'}^t \gamma_1(s)\,\mathrm{d}s} Z(t')\,\mathrm{d}t'\right]\right) \tag{3.2}$$

where k_0 is a functional of $Z(t)$ given by

$$k_0 = \int_0^\infty Z(s)\exp\left(-\int_0^s \gamma_1(t)\,\mathrm{d}t\right)\mathrm{d}s \tag{3.3}$$

and the auxiliary functional of noise $\eta(t) \equiv \gamma_2(t)\xi(t)$ is

$$G_\eta([k(t)]) = G_\xi([\gamma_2(t)k(t)]) \tag{3.4}$$

meaning that any m-time moment of the s.p. $\boldsymbol{\eta}(t)$ comes from taking the functional derivative of $G_\xi([\gamma_2(t)k(t)])$ with respect to $k(t)$.

Proof. The proof is entirely analogous to that of proposition 1. Note that from (3.1) we can define the noise $\eta(t) = \dot{Y} + \gamma_1(t)Y$; $k(t)$ is a test function with finite support, and the functional of the new noise $\eta(t)$ is (3.4) (with $\gamma_2(t)$ a sure function of time). □

A trivial example. Let us assume that the random force in (3.1) is a non-stationary T-periodic Gaussian white noise‡. Thus in (3.1) the simplest model would be to adopt $\gamma_1(t) = \gamma_1 =$ constant, and a random force $\gamma_2(t)\xi(t) = \cos(\omega_0 t)\xi(t)$, where $\xi(t)$ is a zero-mean Gaussian white noise (appendix B). Using proposition 2 all the m-time moments of s.p. $\boldsymbol{Y}(t)$ follow immediately by taking the functional derivative (of the test function $Z(t)$) from

$$G_Y([Z(t)]) = G_\xi\left([\cos(\omega_0 t)\int_t^\infty \mathrm{e}^{\gamma_1(t-s)}Z(s)\,\mathrm{d}s]\right)$$

where $G_\xi([k(t)])$ is given in (B1) with $\langle\xi(t)\xi(s)\rangle = \Gamma_2\delta(t-s)$, and we have used $Y(0) = 0$. Thus it is easy to see that the time-asymptotic correlation function of the s.p. $\boldsymbol{Y}(t)$ is

$$\langle\langle Y(\alpha)Y(\beta)\rangle\rangle_{\text{asymp}}$$
$$= \Gamma_2 e^{-\gamma_1|\alpha-\beta|}\frac{\gamma_1^2 + \omega_0^2 + \gamma_1^2\cos(2\omega_0\min\{\alpha,\beta\}) + \gamma_1\omega_0\sin(2\omega_0\min\{\alpha,\beta\})}{4\gamma_1\left(\gamma_1^2+\omega_0^2\right)}$$

† The function $\gamma_1(t)$ can be related to the variational equation in the context of the Ω-expansion.

‡ A similar non-stationary 2π-periodic stochastic process appears in frequency modulated non-equilibrium systems [14].

showing in the spectrum (susceptibility) the expected resonant behaviour with frequency ω_0. Other noise structures (non-Gaussian) can also be analysed in a similar way.

4. On the generalized Wiener process

Another class of stochastic differential equations which are also very important in non-equilibrium statistical mechanics, are those without inertial effects. Subsequently, when the noise term is a Gaussian white process, this picture leads to the Smoluchoswki equation (in the position phase space) rather than the Kramer–Fokker–Planck equation. This fact opens the question of solving general stochastic process (without inertia) but driven by arbitrary noise. This is a formidable task when the *random force* is non-Gaussian and the potential is nonlinear.

Using the same technique used to prove proposition 1 we can consider a generalized Wiener process, i.e. when the potential is linear. So we obtain the stochastic differential equation

$$\dot{X} = C_1 + \xi(t) \qquad X \in [-\infty, \infty]. \tag{4.1}$$

If $C_1 = 0$ and $\xi(t)$ is a zero-mean Gaussian white noise, $\boldsymbol{X}(t)$ is the Wiener process [9]. If $\xi(t)$ is a dichotomous random force (i.e. a noise with a finite correlation) the s.p. $\boldsymbol{X}(t)$ is no longer Markovian and its complete characterization is an interesting task. These types of stochastic differential equations are particular relevant to discuss transport phenomena with finite velocity and some effort has been put to obtain the 1-time probability distribution of the process [15]. An alternative way of working out this problem is by using the functional calculus, from which we obtain the following proposition.

Proposition 3. The generalized s.p. $\boldsymbol{X}(t)$, with a sure initial condition $X(0) = X_0$ is completely characterized by the functional

$$G_X([Z(t)]) = \mathrm{e}^{+\mathrm{i}k_0X_0}\ G_\eta\left(\left[k_0 - \int_0^t Z(s)\,\mathrm{d}s\right]\right) \tag{4.2}$$

where k_0 is a functional of $Z(t)$ given by

$$k_0 = \int_0^\infty Z(s)\,\mathrm{d}s \tag{4.3}$$

and the auxiliary functional of noise $\eta(t)$ is

$$G_\eta([k(t)]) = \exp\left(\mathrm{i}C_1\int_0^\infty k(t)\,\mathrm{d}t\right)G_\xi([k(t)]). \tag{4.4}$$

Proof. The proof is entirely analogous to that of proposition 1. Note that $k(t)$ is a test function with finite support so $k(\infty) = 0$, and the functional of any shifted noise $\eta(t) \equiv C_1 + \xi(t)$ with sure constant C_1 is (4.4). □

The time-dependent generalized Wiener process can also be solved in an analogous way as in proposition 2.

Applications. If in (4.1) $\xi(t)$ is a dichotomous noise (see the functional (C7)) by proposition 3 any m-time moment follows by taking the functional derivative of $G_X([Z(t)])$.

For example the 1-time moment is

$$\langle X(\alpha)\rangle = \mathrm{i}^{-1}\frac{\delta}{\delta Z(\alpha)}G_X([Z(t)])\bigg|_{Z=0} = X_0 + C_1\alpha - \frac{\xi_0}{2\lambda}[\mathrm{e}^{-2\lambda\alpha} - 1] \tag{4.5}$$

where as before ξ_0 is the initial condition of the dichotomous noise. The 2-time moment is

$$\langle X(\alpha)X(\mu)\rangle = -\frac{\delta}{\delta Z(\alpha)}\frac{\delta}{\delta Z(\mu)}G_X([Z(t)])\Big|_{Z=0}$$
$$= (X_0+C_1\alpha)(X_0+C_1\mu)+\frac{(X_0+C_1\alpha)\xi_0}{2\lambda}[1-\mathrm{e}^{-2\lambda\mu}]$$
$$+\frac{(X_0+C_1\mu)\xi_0}{2\lambda}[1-\mathrm{e}^{-2\lambda\alpha}]+\frac{a^2}{\lambda}\left\{\min(\alpha,\mu)+\frac{\mathrm{e}^{-2\lambda\min(\alpha;\mu)}-1}{2\lambda}\right\}. \quad (4.6)$$

Thus a dichotomous noise only changes the transient behaviour, i.e. after a time $t\simeq\max(\alpha,\mu)\gg\frac{1}{2}\lambda$ the second moment (when $C_1=0$) behaves like a Wiener particle $\langle X(t)^2\rangle\sim\frac{a^2}{\lambda}t$. Equation (4.6), in the limit $\alpha=\mu$, is in agreement with the results found by Morita [5] and Hongler [15] who found the 1-time probability distribution (with a stationary dichotomous noise). As the s.p. $\boldsymbol{X}(t)$ is non-Markovian the 1-time probability distribution is not sufficient to completely characterize the process, so that higher order m-time moments are required, which of course can easily be obtained from proposition 3 and for different noise structures.

As a matter of fact, any m-time moment of the s.p. $\boldsymbol{X}(t)$ can be found from our generalized Ornstein–Uhlenbeck process by putting $\gamma=0$ and noting that the constant C_1 only introduces a dynamical redefinition, i.e. from (4.2) to (4.4) the initial condition is shifted to $X_0\to X_0+C_1t$.

For example (when $C_1=0$, and $X_0=0$) the long-time limit of the second cumulant is

$$\langle\langle X(\mu)X(\alpha)\rangle\rangle = 4\rho A^4\tau_c^2\min(\mu,\alpha).$$

here we have used a Campbell's noise with a shape $\psi(t)=A^2\exp(-|t|/\tau_c)$. However, with radioactive decay noise the second cumulant of the generalized Wiener process gives

$$\langle\langle X(\mu)X(\alpha)\rangle\rangle = \xi_0\mathrm{e}^{-\beta\min(\mu,\alpha)}\mu\alpha-\frac{\xi_0}{\beta^2}(1-\mathrm{e}^{-\beta\mu})(1-\mathrm{e}^{-\beta\alpha})$$
$$+\xi_0\frac{\mathrm{e}^{-\beta\min(\mu,\alpha)}}{\beta^2}[-2+2\mathrm{e}^{\beta\min(\mu,\alpha)}-2\beta\min(\alpha,\mu)-\beta^2\min(\alpha,\mu)^2].$$

5. The generalized Kubo oscillator

Narrowing phenomena in magnetic resonance spectra have been studied, in a very elegant way, by a simple oscillator with modulated (random) frequency [10]. Since then several applications of this model have been used in very different problems [9, 16]. Let us now consider the Kubo oscillator, i.e. the stochastic differential equation

$$\dot{U}=(-\gamma+\xi(t))U \quad (5.1)$$

where γ is the dissipation parameter. If $\xi(t)$ is a Gaussian white-noise (5.1) can be reduced (after the introduction of some specific stochastic calculus) to Fokker–Planck dynamics. We propose here to characterize the s.p. $\boldsymbol{U}(t)$ when the noise $\xi(t)$ is an arbitrary process characterized by its functional $G_\xi([k(t)])$.

Unfortunately we cannot give a closed expression for the functional of s.p. $\boldsymbol{U}(t)$, nevertheless all its 1-time moments can be found. Note that knowing all the moments $\langle U(t_1)^p\rangle$ allows us to find the 1-time conditional probability distribution $P[U(t_1)|U(t_0)]$ as a series expansion [17]

$$P[U(t_1)|U(t_0)]\equiv P(U_1,t_1|U_0,t_0)=\left[\sum_{p=0}^{\infty}\frac{1}{p!}\left(-\frac{\partial}{\partial U_0}\right)^p\mathcal{M}_p(t,t_0,U_0)\right]\delta(U-U_0) \quad (5.2)$$

where $\mathcal{M}_p(t_1, t_0, U_0) \equiv \langle (U(t_1) - U_0)^p \rangle$ are the centred moments which can be calculated from proposition 4 (or 5) below and by employing Newton's binomial.

The 1-time moments of the s.p. $\boldsymbol{U}(t)$ can be found by

Proposition 4.

$$\langle U(t_1)^p \rangle = U_0^p \mathrm{e}^{-\gamma p t_1} G_\xi([k(t) = -\mathrm{i} p \Theta(t_1 - t)]) \tag{5.3}$$

where $\Theta(t)$ is the step function† and U_0 the initial condition.

Proof. The proof is accomplished by introducing the change of variable $H = \ln U$ in (5.1) and noting that the functional of the s.p. $\boldsymbol{H}(t)$ reduces to the generalized Wiener case with a shifted noise $\eta(t) \equiv -\gamma + \xi(t)$, so using proposition 3 (with $C_1 = -\gamma$) we obtain

$$G_H([Z(t)]) = \mathrm{e}^{\mathrm{i}k_0 H_0} G_\eta\left([k_0 - \int_0^t Z(s)\,\mathrm{d}s]\right)$$

with $k_0 = \int_0^\infty Z(s)\,\mathrm{d}s$, $H_0 = \ln U_0$ and $U_0 \equiv U(0)$ is the sure initial condition of s.p. $\boldsymbol{U}(t)$. Noting that $\langle U^p \rangle = \langle \mathrm{e}^{pH} \rangle$ the $p-$moments of s.p. $\boldsymbol{U}(t)$ are given in terms of the characteristic function $G_{H(t_1)}(k) \equiv \langle \exp(\mathrm{i}kH(t_1)) \rangle$ evaluated at $k = -\mathrm{i}p$. Using the fact that the characteristic function of $H(t_1)$ is obtained from the characteristic functional $G_H([Z(t)])$ and by putting the test function $Z(t) = k\delta(t - t_1)$ we obtain

$$G_{H(t_1)}(k) = G_H([Z(t) = k\delta(t - t_1)]).$$

Therefore integrating, using the definition of the step function $\Theta(t)$ and the fact that $G_\eta([k(t)]) = \mathrm{e}^{-\mathrm{i}\gamma \int_0^\infty k(t)\,\mathrm{d}t} G_\xi([k(t)])$, proposition 4 follows immediately. □

Application. Giving any functional of noise $G_\xi([k(t)])$ proposition 4 allows us to obtain all the 1-time moments of s.p. $\boldsymbol{U}(t)$. Trivial examples follow for the Gaussian noise, etc (see appendix B).

With Campbell's noise the 1-time moments are

$$\langle U(t_1)^p \rangle = U_0^p \mathrm{e}^{-\gamma p t_1} \exp \int_0^\infty \mathrm{d}\tau\, q(\tau) \left[\exp p \int_0^{t_1} \psi(s - \tau)\,\mathrm{d}s - 1 \right]. \tag{5.4}$$

In the particular case when $q(s)$ is uniform, and $\psi(t) = B\delta(t)$ (shot noise) the moments can be written as

$$\langle U(t_1)^p \rangle = U_0^p \exp t_1[\rho(\mathrm{e}^{pB} - 1) - p\gamma]. \tag{5.5}$$

This shows an exponential behaviour with p which is quite different from the (zero-mean) Gaussian white-noise case: $\langle U(t_1)^p \rangle = U_0^p \mathrm{e}^{-\gamma p t_1} \exp(\frac{p^2 \Gamma_2}{2} t_1)$. From (5.5) we see that the p-moment diverges only if $\mathrm{e}^{pB} > 1 + p\gamma/\rho$, thus if B (the amplitude of the shot noise) fulfils $B > \gamma/\rho$ all the moments of generalized Kubo oscillator will diverge.

If we use a stationary dichotomous noise the 1-time moments are

$$\langle U(t_1)^p \rangle = U_0^p \mathrm{e}^{-(\lambda + \gamma p) t_1} \left[\cosh\left(\sqrt{\lambda^2 + p^2 a^2} t_1\right) + \frac{\lambda}{\sqrt{\lambda^2 + p^2 a^2}} \sinh\left(\sqrt{\lambda^2 + p^2 a^2} t_1\right) \right]. \tag{5.6}$$

To obtain this result we have used proposition 4 and the fact that from proposition 3 all the 1-time moments of the generalized Wiener process can be found. Thus we can use

† Note that, the way we are using the differential calculus corresponds to the Stratonovich interpretation of (5.1).

the characteristic function from Morita's paper [5], which is in agreement with our 1-time moments. We see from (5.6) that the $p-$moment diverges only if $p(a^2-\gamma^2) > 2\gamma\lambda$. Note that if $\gamma > a$ (the amplitude of the dichotomous noise) none of the moments will diverge.

If we use a radioactive decay noise the 1-time moments are

$$\langle U(t_1)^p\rangle = U_0^p \mathrm{e}^{-\gamma p t_1}\left[\beta\frac{1-\mathrm{e}^{-t_1(\beta-p)}}{(\beta-p)} + \mathrm{e}^{-t_1(\beta-p)}\right]^{\xi_0}. \tag{5.7}$$

If $\beta = p$ the moment has a different behaviour: $\langle U(t_1)^\beta\rangle = U_0^\beta \mathrm{e}^{-\gamma\beta t_1}[\beta t_1+1]^{\xi_0}$. We see from (5.7) that for $(\beta-p)<0$ the moments diverge only if $\gamma < \xi_0(p-\beta)/p$. Note that here β is compared with a natural number because the jump size of the noise is one, this fact can be seen from the characteristic functional (2.20). Due to the fact that the radioactive decay noise is non-stationary all these moments depend on the initial condition ξ_0.

Discussion. Depending on the noise and the value of dissipation parameter γ, the evolution of $\langle U(t_1)^p\rangle$ will be different; this fact is strongly dependent on the noise amplitude. From these simple results we conclude that noise's amplitude controls (in a non-trivial way) the time-evolution of the moments. This situation depends on the structure of the noise, on the other hand this fact does not appear for non-multiplicative stochastic differential equations.

The complex Kubo case, i.e. $\dot{U} = -\mathrm{i}(\omega_0+\xi(t))U$ with $U \to U_1 + \mathrm{i}U_2$; can trivially be obtained from our previous results by defining an imaginary noise $\xi \to -\mathrm{i}\xi$ and dissipation $\gamma \to \mathrm{i}\omega_0$. For example, for the complex Kubo oscillator with Campbell's noise we obtain

$$\langle U(t_1)^p\rangle = U_0^p \mathrm{e}^{-\mathrm{i}\omega_0 p t_1}\exp(t_1\rho(\mathrm{e}^{-\mathrm{i}pB}-1)). \tag{5.8}$$

However, with a dichotomous noise we obtain:

$$\langle U(t_1)^p\rangle = U_0^p \mathrm{e}^{-(\lambda+\mathrm{i}\omega_0 p)t_1}\left[\cos\left(\sqrt{p^2a^2-\lambda^2}t_1\right) + \frac{\lambda}{\sqrt{p^2a^2-\lambda^2}}\sin\left(\sqrt{p^2a^2-\lambda^2}t_1\right)\right]. \tag{5.9}$$

With a radioactive decay noise the moments are

$$\langle U(t_1)^p\rangle = U_0^p \mathrm{e}^{-\mathrm{i}\omega_0 p t_1}\left[\beta\frac{1-\mathrm{e}^{-t_1(\beta+\mathrm{i}p)}}{(\beta+\mathrm{i}p)} + \mathrm{e}^{-t_1(\beta+\mathrm{i}p)}\right]^{\xi_0}. \tag{5.10}$$

Showing, in all cases, that the moments have an oscillatory decaying behaviour. From these results the spectral analysis of the moments follow in a straightforward manner.

5.1. The time-dependent case

Let us generalize (5.1) to the stochastic differential equation:

$$\dot{U} = -\gamma_1(t)U + U\gamma_2(t)\xi(t) \tag{5.11}$$

where, as before, the *random force* $\xi(t)$ is an arbitrary noise completely characterized by its functional $G_\xi([k(t)])$, $\gamma_2(t)$ is a sure function of time and $\gamma_1(t)$ is any positive function of time. Thus 1-time moments of s.p. $\boldsymbol{U}(t)$ follow.

Proposition 5.

$$\langle U(t_1)^p\rangle = U_0^p \exp\left(-p\int_0^{t_1}\gamma_1(s)\,\mathrm{d}s\right)G_\xi([k(t) = -\mathrm{i}p\gamma_2(t)\Theta(t_1-t)]). \tag{5.12}$$

Proof. Using a procedure similar to the one used to prove proposition 4 we introduce the s.p. $\boldsymbol{H}(t)$, which is now a time-dependent generalized Wiener one. This s.p. $\boldsymbol{H}(t)$ can be solved in an entirely analogous way as in proving proposition 2. Therefore the 1-time moments of s.p. $\boldsymbol{U}(t)$ (5.12) follow immediately. □

5.2. Discussion on nonlinear models

Other (particular) types of nonlinear stochastic differential equation can also be obtained in a similar way.

Example 1. Let the stochastic differential equation be

$$\dot{Q} = -\gamma_1(t) Q \ln Q + Q\gamma_2(t)\xi(t) \qquad Q \in [0, \infty] \tag{5.13}$$

where $\xi(t)$ is an arbitrary noise characterized by its functional $G_\xi([k(t)])$, $\gamma_1(t)$ and $\gamma_2(t)$ are non-random functions as before. Introducing the change of variable $Y = \ln Q$ we obtain the generalized Ornstein–Uhlenbeck process (3.1). The s.p. $\boldsymbol{Y}(t)$ was completely solved by proposition 2, thus all the 1-time moments of s.p. $\boldsymbol{Q}(t)$ can be calculated as $\langle Q(t_1)^p\rangle = \langle \exp pY(t_1)\rangle$. So we obtain

$$\langle Q(t_1)^p\rangle = G_Y([Z(t) = -\mathrm{i}p\delta(t - t_1)]).$$

Example 2. Let the stochastic differential equation be

$$\dot{Q} = (-\gamma_1(t) + \gamma_2(t)\xi(t)) Q \ln Q \qquad Q \in [0, \infty] \tag{5.14}$$

here $\gamma_1(t)$, $\gamma_2(t)$ and $\xi(t)$ are as in example 1. Introducing the change of variable $U = \ln Q$ we reduce (5.14) to the generalized Kubo process (5.11). From proposition 5 we can calculate the moments $\langle U(t_1)^p\rangle$, therefore all the 1-time moments of s.p. $\boldsymbol{Q}(t)$ are given as

$$\langle Q(t_1)^p\rangle = \sum_{j=0}^{\infty} \frac{p^j}{j!} \langle U(t_1)^j\rangle.$$

We see that both examples have been reduced with the help of our previous propositions.

6. Conclusions

The generalized Ornstein–Uhlenbeck s.p. $\boldsymbol{V}(t)$—with natural boundary conditions—was completely characterized in terms of its functional $G_V([Z(t)])$, i.e. proposition 1. Several models of noises have been constructed, particular stress has been put in the Campbell, dichotomous, and radioactive decay noises $\xi(t)$. This fairly general method is based upon knowing the characteristic functional of the noise $G_\xi([k(t)])$. Thus any m-time moment of the s.p. $\boldsymbol{V}(t)$ follows simply by taking the functional derivative of $G_V([Z(t)])$. One of the questions addressed in this paper was the characterization of the linear relaxation in a generalized medium (different noise structures). In general the medium was represented by a friction term γ and an arbitrary random force $\xi(t)$ (additive noise) in a Langevin-like equation (2.1). We have exemplified the method with the calculation of linear-relaxation mechanisms (slowing down), correlation functions (susceptibilities follow simply), and diffusion constants.

The generalized Ornstein–Uhlenbeck process (3.1), when the coefficients are time-dependent, has also been worked out in proposition 2. The complete characterization has been given in terms of its functional. This formulation is exact and provides a systematic starting point to obtain higher-order cumulants and also to compute other non-trivial objects.

As a final remark on the generalized Ornstein–Uhlenbeck process we would like to comment that the interesting phenomenon of the rotation of a molecule in interaction with its neighbours can be studied by using the results of this approach. The statistical properties of the random torques can be very different from the usual Gaussian assumptions [18]. Thus it is worth studying the simplest relaxation of a plane rotation characterized by the stochastic differential equation $\ddot{\theta} + \gamma\dot{\theta} = \xi(t)$, where the quantity of interest is the cosine of the polar angle $\cos\theta(t)$. It is possible to see that a nontrivial object such as $\langle\langle\cos[\theta(t_1) - \theta(t_2)]\rangle\rangle$ can be calculated by using proposition 1 (or 2 if $\gamma \to \gamma(t)$). The ideal hard-collision approximation (Gaussian random torques) can be generalized by considering different noise structures—as the one presented in this paper—working along these lines is in progress.

The generalized Wiener s.p. $\boldsymbol{X}(t)$—with natural boundary conditions—has been defined in (4.1) and its complete characterization was achieved by proposition 3. We have worked out the s.p. $\boldsymbol{X}(t)$ when $\boldsymbol{\xi}(t)$ is any s.p. characterized by its functional $G_\xi([k(t)])$; some particular examples of noises have been shown.

The so-called Kubo oscillator $\boldsymbol{U}(t)$ with arbitrary noise $\xi(t)$ has been studied (5.1), and all the 1-time moments have been characterized by proposition 4. Examples follow by using Campbell's, dichotomous, and radioactive decay noises. The case (5.11) with time-dependent coefficients has also been worked out, and all the 1-time moments characterized by proposition 5. A closed expression for the 1-time conditional probability distribution can be found by quadrature, i.e. proposition 4 (or 5) allows us to write a series expansion for the probability (5.2).

On the other hand the interesting phenomenon of the 'broadening line' in the complex Kubo oscillator (for different noises) can also be studied from results (5.8) to (5.10). Note that proposition 4 (or 5) allows us to find a closed expression for all the 1-time moments of the Kubo oscillator, in contrast to previously reported methods [10].

We remark that there is no limitation on the calculation of any higher-order cumulant of our generalized Orsntein-Uhlenbeck process. When the functional of noise is only known as a series expansion the calculation follows diagrammatically as was shown in appendix A. Some mathematical details about the functional of simple noises can be found in appendix B. In general the functional $G_\xi([k(t)])$ of any Markov noise can be found using the Darling–Siegert theorem. Thus in appendix C we have used this theorem to calculated the characteristic functional of the dichotomous noise.

In this paper we have used a functional technique to solve problems with natural boundary conditions, the application of this method to problems with non-natural boundary condition is under investigation.

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Appendix A. Cluster properties for the cumulants

The problem of solving all the cumulants, if we know all the moments, is a very well known task [9, 10]. In order to make this paper self-contained, and following proposition 1, we want to give here a simple diagrammatic technique from which all the cumulants can be drawn as cluster diagrams, i.e. any m-time cumulant can be drawn in terms of the m-time moment and all the previous n-time cumulants (with $n < m$). Figure 1 shows some diagrams of cumulants. From these graphs it is possible to see that their construction follows certain prescriptions.

(1) For each random variable we introduce a vertex $\bullet$ into the graph.

(2) Each double bond represents a cumulant between the connecting vertexes.

(3) Each simple bond corresponds to a moment between the connecting vertexes.

(4) Each isolated vertex $\odot$ represents the first moment of the vertex.

(5) Any m-time cumulant is built up by subtracting from the m-time moment all the different graphs involving all the possible n-time cumulants of lower order, i.e. $n < m$.

From (2.3) it is possible to write down all the cumulants of our generalized velocity process if we know the functional of the noise $G_\xi([k(t)])$ and we take functional derivatives of $\ln G_V([Z(t)])$, i.e. using proposition 1. Note that the diagrammatic technique is a useful tool when the characteristic functional of the noise is only known as a series expansion, as in (C7).

Appendix B. On the characteristic functional of simple noises

If $\xi(t)$ is a zero-mean Gaussian non-white noise, with $t \in [0, \infty]$ the characteristic functional can be found by using the cumulant expansion technique [9]

$$G_\xi([k(t)]) = \left\langle \exp \int_0^\infty \mathrm{i}k(t)\xi(t)\,\mathrm{d}t \right\rangle = \exp \frac{-1}{2} \int_0^\infty \int_0^\infty k(s)k(t)\langle\xi(t)\xi(s)\rangle \mathrm{d}t\,\mathrm{d}s. \tag{B1}$$

If $\xi(t)$ is a stationary non-Gaussian white noise, i.e. for arbitrary constants Γ_n we have the following cumulants (for $n \geqslant 2$)

$$\langle\langle \xi(t_1)\xi(t_2)\xi(t_3)\ldots\xi(t_n)\rangle\rangle = \Gamma_n\ \delta(t_1 - t_2)\delta(t_1 - t_3)\ldots\delta(t_1 - t_n). \tag{B2}$$

Then with $t \in [0, \infty]$ the characteristic functional of s.p. $\boldsymbol{\xi}(t)$ can be found as [19]

$$G_\xi([k(t)]) = \exp \sum_{m=1}^{\infty} \frac{\mathrm{i}^m}{m!} \Gamma_m \int_0^\infty (k(t))^m\,\mathrm{d}t. \tag{B3}$$

For example the Poisson white-noise is a monoparametric s.p. (with $\Gamma_m = \Gamma$), which is completely characterized by the functional $G_\xi([k(t)]) = \exp(\Gamma \int_0^\infty \mathrm{d}t\,[\exp \mathrm{i}k(t) - 1])$. So all the cumulants of the s.p. $\boldsymbol{V}(t)$ (with Poisson random forces) would follow just by using (2.3) and taking the functional derivative of $\ln G_V([Z(t)])$.

For a generalized white-noise and following from proposition 1, the 2-time moment of s.p. $\boldsymbol{V}(t)$ will only depend on the constants Γ_1 and Γ_2. Only higher correlation functions of s.p. $\boldsymbol{V}(t)$ will carry information on the non-Gaussian structure appearing in (B3), i.e. Γ_n with $n \geqslant 3$.

Appendix C. On the characteristic functional of the dichotomous noise

In section 2.2 we commented that the characteristic functional of the dichotomous noise cannot be summed in a closed way, this is why we wrote only its series expansion in (2.15). Here we show by using the Darling–Siegert theorem [12, 13] why we cannot find a closed expression for such a functional.

The Darling–Siegert theorem states that for any Markov process it is possible write a masterly equation from which the characteristic functional can in principle be solved.

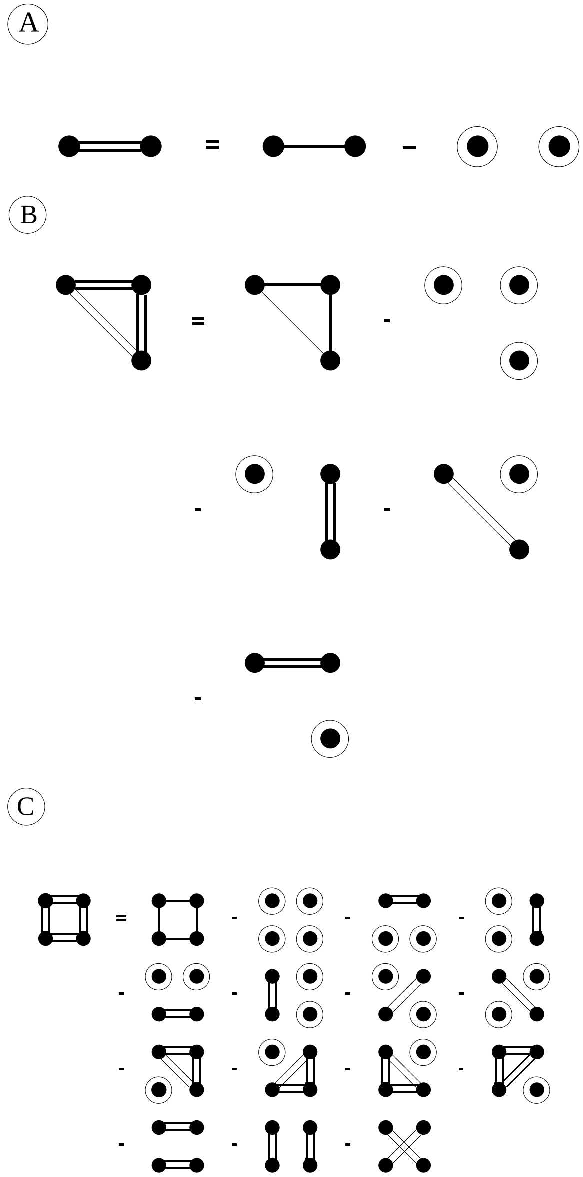

Figure 1. First non-trivial cummulants corresponding to an arbitrary stochastic process X_t. (a) Diagram representing the second cumulant: $\langle\langle X_1X_2\rangle\rangle = \langle X_1X_2\rangle - \langle X_1\rangle\langle X_2\rangle$. ($b$) Diagram representing the third cumulant: $\langle\langle X_1X_2X_3\rangle\rangle = \langle X_1X_2X_3\rangle - \langle X_1\rangle\langle X_2\rangle\langle X_3\rangle - \langle\langle X_1X_2\rangle\rangle\langle X_3\rangle - \langle\langle X_1X_3\rangle\rangle\langle X_2\rangle - \langle\langle X_3X_2\rangle\rangle\langle X_1\rangle$. ($c$) Diagram representing the fourth cumulant: $\langle\langle X_1X_2X_3X_4\rangle\rangle = \langle X_1X_2X_3X_4\rangle - \langle X_1\rangle\langle X_2\rangle\langle X_3\rangle\langle X_4\rangle - \langle\langle X_1X_2\rangle\rangle\langle X_3\rangle\langle X_4\rangle - \langle X_1\rangle\langle\langle X_2X_3\rangle\rangle\langle X_4\rangle - \langle X_1\rangle\langle X_2\rangle\langle\langle X_3X_4\rangle\rangle - \langle\langle X_1X_4\rangle\rangle\langle X_2\rangle\langle X_3\rangle - \langle X_1\rangle\langle\langle X_2X_4\rangle\rangle\langle X_3\rangle - \langle\langle X_1X_3\rangle\rangle\langle X_2\rangle\langle X_4\rangle - \langle\langle X_1X_2X_3\rangle\rangle\langle X_4\rangle - \langle X_1\rangle\langle\langle X_2X_3X_4\rangle\rangle - \langle\langle X_1X_3X_4\rangle\rangle\langle X_2\rangle - \langle\langle X_1X_2X_4\rangle\rangle\langle X_3\rangle - \langle\langle X_1X_2\rangle\rangle\langle\langle X_4X_3\rangle\rangle - \langle\langle X_1X_4\rangle\rangle\langle\langle X_2X_3\rangle\rangle - \langle\langle X_1X_3\rangle\rangle\langle\langle X_2X_4\rangle\rangle$. Any non-conmutative cumulant diagram follows by using prescriptions 1–5 .

Let the Markov process be the 2-state process characterized by (2.14); then the *curtailed characteristic functional* $\mathcal{G}$ is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}\mathcal{G} + \mathrm{i}ak(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathcal{G} \tag{C1}$$

with the initial condition $\mathcal{G}(t = 0) = \boldsymbol{\delta}_{\xi;\xi_0}$. This masterly equation involves a continuous test function $k(t)$ with finite support. Thus for the dichotomous noise the characteristic functional is given by

$$G_\xi([k(t)]) = \lim_{t\to\infty}\sum_{l=1}^{2}\mathcal{G}_l([k(t)], t) \tag{C2}$$

where $\mathcal{G}_l$ represents the lth component of $\mathcal{G}$.

Defining new variables $\eta \equiv \mathcal{G}_1([k(t)], t) + \mathcal{G}_2([k(t)], t)$ and $\chi \equiv \mathcal{G}_1([k(t)], t) - \mathcal{G}_2([k(t)], t)$ equation (C1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = [-2\lambda\mathbf{M} + \mathrm{i}ak(t)\mathbf{N}]\mathcal{W} \tag{C3}$$

where the components of the vector $\mathcal{W}(t)$ are $\mathcal{W}_1 \equiv \mathcal{W}_1([k(t)], t) = \eta$; $\mathcal{W}_2 \equiv \mathcal{W}_2([k(t)], t) = \chi$, and the 2×2 matrices $\mathbf{M}$, $\mathbf{N}$ are given by

$$\mathbf{M} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{N} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{C4}$$

Introducing the matrices

$$\mathbf{A}_0 \equiv -2\lambda\mathbf{M} \qquad \mathbf{A}_1(t) \equiv \mathrm{i}ak(t)\mathbf{N}$$

equation (C3) can be written in the interaction representation as

$$\dot{W} = \boldsymbol{R}(t)W \tag{C5}$$

Where $\boldsymbol{R}(t) \equiv \exp(-t\mathbf{A}_0)\mathbf{A}_1(t)\exp(+t\mathbf{A}_0)$ and $W(t) \equiv \exp(+t\mathbf{A}_0)\mathcal{W}(t)$, therefore from (C5) we obtain for each component of $W(t)$ (in the interaction representation)

$$\begin{pmatrix} \dot{\eta}_I \\ \dot{\chi}_I \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{i}ak(t)\mathrm{e}^{-2\lambda t} \\ \mathrm{i}ak(t)\mathrm{e}^{+2\lambda t} & 0 \end{pmatrix}\begin{pmatrix} \eta_I \\ \chi_I \end{pmatrix}. \tag{C6}$$

An explicit solution of (C6) involves a time-ordering operator, unfortunately it is not possible to obtain a closed expression [20] for $W(t)$ but its series expansion is simple to find.

Using the initial condition for the *curtailed characteristic functional*, in the interaction representation we obtain

$$W(0) = \begin{pmatrix} \delta_{a,\xi_0} + \delta_{-a,\xi_0} \\ \delta_{a,\xi_0} - \delta_{-a,\xi_0} \end{pmatrix}.$$

Thus going back to the first representation, using (C2) and the fact that the characteristic functional is the first component of $\mathcal{G}$ we obtain

$$\begin{aligned} G_\xi([k(t)]) = \Big\{ & 1 - a^2\int_0^\infty \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 k(t_1)k(t_2)\mathrm{e}^{-2\lambda(t_1-t_2)} \\ & + a^4\int_0^\infty \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3\int_0^{t_4}\mathrm{d}t_4 k(t_1)k(t_2)k(t_3)k(t_4)\mathrm{e}^{-2\lambda(t_1-t_2+t_3-t_4)} + \cdots \Big\} \\ & + \mathrm{i}\Big\{ a\int_0^\infty \mathrm{d}t_1 k(t_1)\mathrm{e}^{-2\lambda t_1} - a^3\int_0^\infty \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3 k(t_1) \end{aligned}$$

$$\times k(t_2)k(t_3)\mathrm{e}^{-2\lambda(t_1-t_2+t_3)} + a^5 \int_0^\infty \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_4 \int_0^{t_4} \mathrm{d}t_5 k(t_1)$$
$$\times k(t_2)k(t_3)k(t_4)k(t_5)\mathrm{e}^{-2\lambda(t_1-t_2+t_3-t_4+t_5)} - \ldots \Bigg\} (\delta_{a,\xi_0} - \delta_{-a,\xi_0}) \qquad \text{(C7)}$$

where we have used that $\delta_{a,\xi_0} + \delta_{-a,\xi_0} = 1$. Note that $a^n(\delta_{a,\xi_0} - \delta_{-a,\xi_0}) = (\pm a)^n = (\xi_0)^n$ the initial condition of the dichotomous noise.

From the exact expression (C7) all the $n-$time moments of the dichotomous noise can easily be calculated, for example its correlation function gives the well known result

$$\langle\langle \xi(s_1)\xi(s_2)\rangle\rangle = \langle \xi(s_1)\xi(s_2)\rangle - \langle \xi(s_1)\rangle\langle \xi(s_2)\rangle$$
$$= a^2 \exp(-2\lambda|s_1 - s_2|) - (\xi_0)^2 \exp(-2\lambda(s_1 + s_2)).$$

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